Solution 8.9

(a) We start with the first-order time-dependent perturbation theory result for transition probability

$$P(t) = \frac{1}{\hbar^2}\sum_f \left| \int_{t'=0}^{t'=t} \langle\psi_f|\hat{W}(t')|\psi_i\rangle e^{i\omega_{fi}t'} dt' \right|^2$$
$$= \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \left| \int_{t'=0}^{t'=t} e^{i(\omega_{fi}+\omega)t'} dt' \right|^2 + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \left| \int_{t'=0}^{t'=t} e^{i(\omega_{fi}-\omega)t'} dt' \right|^2$$

which assumes that each scattering process is an independent parallel channel and where $\hbar\omega_{fi} = (E_f - E_i)$.

Performing the integration we have

$$P(t) = \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \left| \frac{e^{i(\omega_{fi}+\omega)t} - 1}{(\omega_{fi}+\omega)} \right|^2 + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \left| \frac{e^{i(\omega_{fi}-\omega)t} - 1}{(\omega_{fi}-\omega)} \right|^2$$

and using the relation $|e^x - 1|^2 = 4\sin^2(x/2)$ gives

$$P(t) = \frac{4W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \frac{\sin^2((\omega_{fi}+\omega)t/2)}{(\omega_{fi}+\omega)^2 t} t + \frac{4W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \frac{\sin^2((\omega_{fi}-\omega)t/2)}{(\omega_{fi}-\omega)^2 t} t$$

(b) The equation in (a) can be re-written as

$$P(t) = \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \frac{\sin^2\left(\left(\frac{(\omega_{fi}+\omega)}{2}\right)t\right)}{\left(\frac{\omega_{fi}+\omega}{2}\right)^2 t} t + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \frac{\sin^2((\omega_{fi}-\omega)t/2)}{\left(\frac{\omega_{fi}-\omega}{2}\right)^2 t} t$$

In the limit $t \to \infty$ we use $\delta(x) = \frac{1}{\pi}\lim_{\eta\to\infty}\frac{\sin^2(\eta x)}{\eta x^2}$. Setting $x = (\omega_{fi}+\omega)/2$ for the first term on the right hand side and $x = (\omega_{fi}-\omega)/2$ for the second term we have

$$P(t) = \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \left(\pi\delta\left(\frac{(\omega_{fi}+\omega)}{2}\right)\right) t + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \left(\pi\delta\left(\frac{(\omega_{fi}-\omega)}{2}\right)\right) t$$

Now, using the fact that $\delta(ax) = \frac{1}{|a|}\delta(x)$ and letting $a = 2\hbar$ so that $2\hbar\delta(\hbar(\omega_{fi} \pm \omega)) = \delta\left(\frac{\omega_{fi} \pm \omega}{2}\right)$, the probability becomes

$$P(t) = \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2 2\hbar\pi t\delta(\hbar(\omega_{fi}+\omega)) + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 2\hbar\pi t\delta(\hbar(\omega_{fi}-\omega))$$

Since $\hbar\omega_{fi} = (E_f - E_i)$ and the scattering rate is $\frac{1}{\tau} = \frac{dP}{dt}$ we have

$$\frac{1}{\tau} = \frac{2\pi}{\hbar} W_0^2 |\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \delta(E_f - E_i + \hbar\omega) + \frac{2\pi}{\hbar} W_0^2 |\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \delta(E_f - E_i - \hbar\omega)$$

in which the first term corresponds to stimulated emission and the second term to absorption of a quanta of energy $\hbar\omega$.

Solution 8.10

(a) The induced transition rate is

$$B = \frac{\pi e^2}{3\varepsilon_0 \hbar^2} \left|\langle j|\hat{\mathbf{r}}|k\rangle\right|^2 U(\omega)$$

where $\langle j|\hat{\mathbf{r}}|k\rangle$ is the matrix element coupling the excited state to the ground state, $\hbar\omega = \Delta E = E_2 - E_1 = 10.2 \text{ eV}$ is the energy of the transition, and

$$U(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

is the photon energy density per unit volume per unit frequency at thermal equilibrium. The spontaneous transition rate is

$$A = \frac{e^2 \omega^3}{3\pi\varepsilon_0 \hbar c^3} \left|\langle j|\hat{\mathbf{r}}|k\rangle\right|^2$$

If the induced and spontaneous transition rates are equal, then

$$\frac{\omega^3}{c^3} = \frac{\pi^2}{\hbar} U(\omega)$$

so that

$$\frac{\omega^3}{c^3} = \frac{\pi^2}{\hbar} \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$$1 = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

and hence

$$T = \frac{\hbar\omega}{k_B \ln(2)} = \frac{10.2}{8.617 \times 10^{-5} \times 0.693} = 1.7 \times 10^5 \text{ K}$$

(b) In the GaAs quantum dot the ground state energy is

$$E_1 = \frac{\hbar^2 \pi^2}{2m^* L^2} = \frac{(1.05 \times 10^{-34})^2 \pi^2}{2 \times 0.07 \times 9.1 \times 10^{-31} \times (20 \times 10^{-9})^2} = 13.3 \text{ meV}$$

and

$$\hbar\omega = \Delta E = E_2 - E_1 = 3 \times E_1 = 40 \text{ meV}$$

Because the dielectric medium has refractive index $n_r = 3.3$ the photon energy density is modified to

$$U(\omega) = \frac{\hbar\omega^3 n_r^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

and, following the steps in (a), we obtain

$$1 = \frac{n_r^3}{e^{\hbar\omega/k_B T} - 1}$$

so that

$$T = \frac{\hbar\omega}{k_B \ln(n_r^3 + 1)} = \frac{0.040}{8.617 \times 10^{-5} \times 3.609} = 129 \text{ K}$$

In our calculations we have assumed that the cavity does not have high Q optical resonances that could either enhance or suppress the emission rate. We also assume that non radiative loss channels such as phonon scattering are not significant.

Solution 8.11

(a) The two-level atom described by Hamiltonian $\hat{H}_0$ has eigenstates $|1\rangle$ and $|2\rangle$ so that

$$\hat{H}_0|1\rangle = E_1|1\rangle$$

and

$$\hat{H}_0|2\rangle = E_2|2\rangle$$

The energy separation is $\hbar\omega_{21} = E_2 - E_1$. The atom is initially in its ground state $|1\rangle$ and at time $t = 0$ it is illuminated with an electric field $\mathbf{E} = |\mathbf{E}_0|(e^{i\omega t} + e^{-i\omega t})$ in the x direction. In this case the change in potential is $\hat{W} = -e|\mathbf{E}_0|\hat{x}(e^{i\omega t} + e^{-i\omega t})$ and the new Hamiltonian is

$$\hat{H} = \hat{H}_0 + \hat{W}$$

(b) Substituting $|x, t\rangle = a_1(t)e^{-i\omega_1 t}|1\rangle + a_2(t)e^{-i\omega_2 t}|2\rangle$ into the time dependent Schrödinger equation $i\hbar\frac{\partial}{\partial t}|x, t\rangle = \hat{H}|x, t\rangle$ we have

$$i\hbar\frac{d}{dt}a_1(t)e^{-i\omega_1 t}|1\rangle + \hbar\omega_1 a_1(t)e^{-i\omega_1 t} + i\hbar\frac{d}{dt}a_2(t)e^{-i\omega_2 t}|2\rangle + \hbar\omega_2 a_2(t)e^{-i\omega_2 t}$$
$$= \hbar\omega_1 a_1(t)e^{-i\omega_1 t}|1\rangle + \hbar\omega_2 a_2(t)e^{-i\omega_2 t}|2\rangle + a_1(t)e^{-i\omega_1 t}\hat{W}|1\rangle + a_2(t)e^{-i\omega_2 t}\hat{W}|2\rangle$$

so that

$$i\hbar\frac{d}{dt}a_1(t)e^{-i\omega_1 t}|1\rangle + i\hbar\frac{d}{dt}a_2(t)e^{-i\omega_2 t}|2\rangle = a_1(t)e^{-i\omega_1 t}\hat{W}|1\rangle + a_2(t)e^{-i\omega_2 t}\hat{W}|2\rangle$$

(c) Multiplying both sides by $\langle 1|$ or $\langle 2|$ we obtain two equations

$$i\hbar\frac{d}{dt}a_1(t) = a_1(t)\langle 1|\hat{W}|1\rangle + a_2(t)e^{-i(\omega_2 - \omega_1)t}\langle 1|\hat{W}|2\rangle$$

$$i\hbar\frac{d}{dt}a_2(t) = a_1(t)e^{i(\omega_2 - \omega_1)t}\langle 2|\hat{W}|1\rangle + a_2(t)\langle 2|\hat{W}|2\rangle$$

where $\langle 2|\hat{W}|1\rangle = -e|\mathbf{E}_0|\langle 2|\hat{x}|1\rangle(e^{i\omega t} + e^{-i\omega t}) = W_{21}(e^{i\omega t} + e^{-i\omega t})$, etc. Hence

$$i\hbar\frac{d}{dt}a_1(t) = a_1(t)W_{11}(e^{i\omega t} + e^{-i\omega t}) + a_2(t)e^{-i\omega_{21}t}W_{12}(e^{i\omega t} + e^{-i\omega t})$$

$$i\hbar\frac{d}{dt}a_2(t) = a_1(t)e^{i\omega_{21}t}W_{21}(e^{i\omega t} + e^{-i\omega t}) + a_2(t)W_{22}(e^{i\omega t} + e^{-i\omega t})$$

where $\omega_{21} = \omega_2 - \omega_1$. If $|\mathbf{E}_0|$ is small then the fast oscillating terms can be neglected and the equations become

$$i\hbar\frac{d}{dt}a_1(t) = a_2(t)W_{12}e^{i(\omega - \omega_{21})t}$$

$$i\hbar\frac{d}{dt}a_2(t) = a_1(t)W_{21}e^{i(\omega_{21} - \omega)t}$$

Eliminating a_1 we get

$$\frac{d^2}{dt^2}a_2(t) - i(\omega_{21} - \omega)\frac{d}{dt}a_2(t) + \frac{W_{12}W_{21}}{\hbar^2}a_2(t) = 0$$

Initial conditions are $a_1(t=0) = 1$ and $a_2(t=0) = 0$ and we have $\frac{d}{dt}a_2(t=0) = \frac{-i}{\hbar}a_1(t=0)W_{21} = \frac{-i}{\hbar}W_{21}$

If $\omega = \omega_{21}$, then

$$\frac{d^2}{dt^2}a_2(t) + \frac{W_{12}W_{21}}{\hbar^2}a_2(t) = 0$$

with solution $a_2(t) = Ae^{iW_{12}t/\hbar} + Be^{-iW_{21}t/\hbar}$. From the initial condition $a_2(t=0) = 0$ which means that $A = -B$ so that $a_2(t) = \frac{-i}{2}(e^{iW_{12}t/\hbar} - e^{-iW_{21}t/\hbar}) = \sin\left(\frac{W_{21}t}{\hbar}\right)$ and the probability that the atom will be in state $|2\rangle$ at time $t > 0$ is

$$|a_2(t)|^2 = \sin^2\left(\frac{W_{21}t}{\hbar}\right)$$

(d) When ω is slightly detuned from ω_{21} we try solution $a_2(t) = Ae^{i\omega_+ t} + Be^{i\omega_- t}$ where $\omega_\pm = \frac{1}{2}((\omega_{21} - \omega) \pm \Lambda)$ and $\Lambda = ((\omega_{21} - \omega)^2 + 4|W_{21}|^2)^{1/2}$. The initial condition for a_2 gives

$$a_2(t) = \frac{-2iW_{21}}{\Lambda}e^{i(\omega_{21} - \omega)t/2}\sin\left(\frac{\Lambda t}{2}\right)$$

and the probability is

$$|a_2(t)|^2 = \frac{4|W_{21}|^2}{\Lambda^2}\sin^2\left(\frac{\Lambda t}{2}\right)$$